AUTOMORPHISMS OF THE FINE GRADING OF $sl(n,\mathbb{C})$ ASSOCIATED WITH THE GENERALIZED PAULI MATRICES

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ABSTRACT. We consider the grading of $sl(n,\mathbb{C})$ by the group Π_n of generalized Pauli matrices. The grading decomposes the Lie algebra into $n^2 - 1$ one–dimensional subspaces. In the article we demonstrate that the normalizer of grading decomposition of $sl(n,\mathbb{C})$ in Π_n is the group $SL(2,\mathbb{Z}_n)$, where $\mathbb{Z}_n$ is the cyclic group of order n.

As an example we consider $sl(3,\mathbb{C})$ graded by Π_3 and all contractions preserving that grading. We show that the set of 48 quadratic equations for grading parameters splits into just two orbits of the normalizer of the grading in Π_3.

1. INTRODUCTION

Among the gradings of reductive Lie algebras over the complex number field and the simultaneous gradings of their representation spaces, by far the most important ones are the gradings by maximal torus. In the case of the Lie algebra it is also called root or Cartan decomposition. Such a grading means a decomposition into eigenspaces of the maximal torus. For a greater part of the past century such gradings have been the workhorses of the theory and applications.

Typical role a Lie algebra plays in physics is the algebra of infinitesimal symmetries of a physical system, which themselves are described in terms elements of representation spaces, eigenvectors of the maximal torus. The corresponding eigenvalues are then the quantum numbers.

The question about existence of other gradings, like those by maximal torus (called fine gradings), has been raised systematically in [1] and solved for the simple Lie algebras of over $\mathbb{C}$ in [2, 3, 4] and recently also for the real number field in [5, 6].

Gradings of Lie algebras are closely related with their automorphisms. In a seminal paper [1] in 1989, it was shown that the finest gradings (called fine) of finite-dimensional simple Lie algebras $\mathcal{L}$ can be classified (up to equivalence generated by elements of Aut $\mathcal{L}$) by the Maximal Abelian groups of Diagonable automorphisms of $\mathcal{L}$, briefly the MAD–groups. In general, the MAD–groups are composed, besides subgroups of the maximal torus, by well–known outer automorphisms [7], and by elements of finite order (EFO's) in the corresponding Lie groups. Since the conjugacy classes of EFO's were systematically described in [8] (see also [7]), it was not difficult to classify the fine gradings in the lowest cases like $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$ [9, 10, 11].

A prominent role in the grading problem of simple Lie algebras is played by the finite group Π_n of n^3 matrices $\mathbb{C}^{n \times n}$. A subset of Π_n, consisting of $n^2 - 1$ traceless matrices, can be taken as a basis of $sl(n,\mathbb{C})$. Since these traceless matrices of are

used as a basis of an Lie algebra, they can be normalized in any convenient way. In particular, for $n = 2$ the Pauli matrices of $sl(2, \mathbb{C})$ are obtained.

Most importantly, the adjoint action of Π_n, though non–Abelian in general, induces an Abelian action on $sl(n, \mathbb{C})$. The fine grading of $sl(n, \mathbb{C})$, which arises in this way, decomposes the algebra into one–dimensional subspaces generated by traceless elements of Π_n [9].

In further prospect, the gradings are to be used e.g. for constructing of grading preserving contractions (*graded contractions*) of semisimple $\mathcal{L}$ [12, 13]. For gradings involving a decomposition into a small number of grading subspaces, this is a relatively easy task [14, 15, 16]. However for fine gradings of algebras with ranks ≥ 3, the system of quadratic equations for contraction parameters one needs to solve, often gets quite large. The task of solving of such a system, would be simplified by the knowledge of its symmetries. Symmetries that are available are provided by those elements of $\operatorname{Aut} \mathcal{L}$ which leave the given grading invariant.

The main goal of this paper is to demonstrate that the decomposition of $sl(n, \mathbb{C})$, as the fine grading by Π_n, is preserved by the finite group $SL(2, \mathbb{Z}_n)$, acting through its n-dimensional representation. Here $\mathbb{Z}_n$ is the cyclic group of order n. We also illustrate an application of this fact.

Special role of matrices Π_n has been recognized in the physical literature for a long time [17]. In more recent years there was a number of papers where the matrices were used as a basic part of the formalism in the development of quantum mechanics in discrete spaces $\mathbb{Z}_n$. The finite group Π_n plays here the role of the discrete Weyl group acting in an n–dimensional complex Hilbert space. See [18, 19] and references quoted there.

The Π_n-grading of $sl(n, \mathbb{C}$ has other special properties. Let us name just two:

(1) All generators are in the same conjugacy class of $SL(n, \mathbb{C})$. Considered as group elements, they are of order n, belonging to the Costant conjugacy class of finite order elements, specified as $[1, 1, \ldots, 1]$ in the notation introduces in [8]

(2) The Π_n-grading makes explicit the decompostion of $sl(n, \mathbb{C})$ into the sum of $n + 1$ Cartan subalgebra. Indeed, if the element of $\mathcal{P}_n$ defined by a couple (a, b) (a, b considered $\mod n$), belongs to one such Cartan subalgebra, that subalgebra is then generated by the $n - 1$ elements carrying labels $(a, b), (2a, 2b), \ldots, ((n-1)a, (n-1)b)$. Clearly such elements commute and have a non-zero determinant.

In Section 2, the role of automorphisms of the Lie algebra in its gradings is recalled. Grading groups of $sl(n, \mathbb{C})$ which do not involve outer automorphisms are described in Section 3. Our main result is in Section 4, namely the normalizers of the grading groups. Section 5 contains an application to $sl(3, \mathbb{C})$: It is shown that the set of 48 quadratic equations for contraction parameters splits into just two orbits of the normalizer of the grading group Π_3.

2. Gradings and automorphisms of Lie algebras

A grading of Lie algebra $\mathcal{L}$ is a decomposition of $\mathcal{L}$ into direct sum of subspaces

$$\Gamma : \quad \mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i \tag{1}$$

such that for any pair of indices $i, j \in I$ there exists a index $k \in I$ with the property

$$[\mathcal{L}_i, \mathcal{L}_j] := \{[X, Y] \mid X \in \mathcal{L}_i, Y \in \mathcal{L}_j\} \subseteq \mathcal{L}_k$$

A grading which cannot be further refined is called *fine*.

Gradings can be obtained by looking at Aut $\mathcal{L}$, the group of all automorphisms of $\mathcal{L}$. It consists of all non–singular linear transformations ϕ of $\mathcal{L}$ as linear space ($\phi \in \mathrm{GL}(\mathcal{L})$) which preserve the binary operation in $\mathcal{L}$:

$$\phi[X, Y] = [\phi X, \phi Y].$$

If ϕ is *diagonable* and X, Y are its eigenvectors with non–zero eigenvalues λ, μ,

$$\phi X = \lambda X, \quad \phi Y = \mu Y,$$

then clearly

$$\phi[X, Y] = [\phi X, \phi Y] = \lambda\mu[X, Y]. \tag{2}$$

This means that the element $[X, Y]$ is either an eigenvector of ϕ with eigenvalue $\lambda\mu$, or is the zero element. The given automorphism ϕ thus leads to a decomposition of the linear space $\mathcal{L}$ into eigenspaces of ϕ,

$$\mathcal{L} = \bigoplus_{i \in I} \mathrm{Ker}(\phi - \lambda_i \, \mathrm{id}),$$

which, according to (2), satisfies the definition of a grading.

Refinements of a given grading, i.e. further decompositions of the subspaces, can be obtained by adjoining further automorphisms commuting with ϕ. Hence, in general, sets $\phi_1, ..., \phi_m$ of mutually commuting automorphisms determine gradings.

Conversely, if a grading (1) of a simple Lie algebra $\mathcal{L}$ is given, it defines a particular Abelian subgroup Diag $\Gamma \subset \mathrm{Aut}\mathcal{L}$ consisting of those automorphisms $\phi \in GL(\mathcal{L})$ which

(i) preserve Γ, $\quad \phi(\mathcal{L}_i) = \mathcal{L}_i$,

(ii) are diagonal, $\phi X = \lambda_i X \quad \forall X \in \mathcal{L}_i$, $i \in I$, where $\lambda_i \neq 0$ depends only on ϕ and $i \in I$.

In [1] an important theorem was proved:

> **Theorem 1.** *Let $\mathcal{L}$ be a finite–dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then the grading Γ is fine, if and only if the diagonal subgroup $Diag\,\Gamma$ is a maximal Abelian group of diagonable automorphisms (shortly MAD–group).*

A general algorithm to construct all MAD–groups for the class of simple classical Lie algebras over complex numbers was given in [2, 3, 4]. [1] These Lie algebras are Lie subalgebras of $gl(n, \mathbb{C})$, hence their MAD–groups can be determined from the MAD–groups of $gl(n, \mathbb{C})$ by imposing certain conditions.

The automorphisms of $gl(n, \mathbb{C})$ can be easily written as combinations of inner and outer automorphisms. For all $X \in gl(n, \mathbb{C})$,

inner automorphisms have the general form

$$\mathrm{Ad}_A X = A^{-1}XA \quad \text{for any} \quad A \in GL(n, \mathbb{C});$$

outer automorphims have the general form

$$\mathrm{Out}_C X = -(C^{-1}XC)^T = \mathrm{Out}_I \mathrm{Ad}_C X, \quad \text{where} \quad C \in GL(n, \mathbb{C}).$$

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[1] Further results concerning the real forms can be found in [5, 6].

Relevant properties of inner and outer automorphisms of $gl(n, \mathbb{C})$ are summarized in the following lemma [2] which allows to express MAD–groups in Aut $gl(n, \mathbb{C})$ in terms of special elements of $GL(n, \mathbb{C})$:

Lemma 2. *Let $A, B, C \in GL(n, \mathbb{C})$.*

(1) *Ad_A is diagonable automorphism if and only if the corresponding matrix A is diagonable.*

(2) *Inner automorphisms commute, $Ad_A \, Ad_B = Ad_B \, Ad_A$, if and only if there exists $q \in \mathbb{C}$ such that*

$$AB = qBA, \quad where \, q \, satisfies \quad q^n = 1. \tag{3}$$

(3) *Out_C is diagonable if and only if $C(C^T)^{-1}$ is diagonable.*

(4) *Inner and outer automorphisms commute, $Ad_A \, Out_C = Out_C \, Ad_A$, if and only if $ACA^T = rC$; since $Ad_{\alpha A} = Ad_A$ for $\alpha \neq 0$, number r can be normalized to unity.*

Remark 3. The sets of complex $n \times n$ matrices satisfying (3) were to our knowledge first studied by H. Weyl [17].

3. MAD–GROUPS WITHOUT OUTER AUTOMORPHISMS

In this contribution we are going to look at the MAD–groups in Aut $gl(n, \mathbb{C})$ without outer automorphisms, i.e. generated by inner automorphisms (the Ad–action in $GL(n, \mathbb{C})$ only). It is shown in [2] that there exists a one-to-one correspondence between MAD-groups without outer automorphisms and Ad-groups (to be defined below) in $GL(n, \mathbb{C})$.

Definition 4. *A subgroup of diagonable matrices $G \subset GL(n, \mathbb{C})$ will be called an **Ad–group** if*

(1) *for any pair $A, B \in G$ the commutator $q(A, B) = ABA^{-1}B^{-1}$ lies in the centre $Z = \{\alpha I_n | \alpha \in \mathbb{C}^*\} \subset GL(n, \mathbb{C})$;*

(2) *G is maximal, i.e. for each $M \notin G$ there exists $A \in G$ such that $q(A, M) \notin Z$.*

In order to describe Ad–groups we introduce the following notation. The subgroup of $GL(n, \mathbb{C})$ containing all regular diagonal matrices will be denoted by $D(n)$. We also define special $k \times k$ diagonal matrices [2]

$$Q_k = \mathrm{diag}(1, \omega_k, \omega_k^2, \ldots, \omega_k^{k-1}),$$

where ω_k is the primitive k–th root of unity, $\omega_k = \exp(2\pi i/k)$, and

$$P_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The unitary matrices P_k, Q_k appear in the finite–dimensional quantum mechanics (FDQM) [17, 18, 19, 20], where their integral powers play the role of exponentiated operators of position and momentum in the position representation. The matrices P_k, Q_k satisfy the identity (3) with $q = \omega_k$,

$$P_k Q_k = \omega_k Q_k P_k, \tag{4}$$

which in FDQM replaces the usual Heisenberg commutation relations. The discrete subgroup of $GL(k, \mathbb{C})$ generated by powers of P_k, Q_k is the *discrete Weyl (or Heisenberg) group* of FDQM in k–dimensional Hilbert space $\mathcal{H}_k$. In [9] this group was called the **Pauli group**; it consists of k^3 elements

$$\Pi_k = \{\omega_k^l Q_k^i P_k^j | i, j, l = 0, 1, \ldots, k-1\}. \tag{5}$$

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[2]For $k = 1$ we set $Q_1 = P_1 = 1$.

The classification of Ad–groups in $GL(n,\mathbb{C})$ is given by the following theorem [3]

Theorem 5. $G \subset GL(n,\mathbb{C})$ *is an Ad–group if and only if* G *is conjugated to one of the finite groups*

$$\Pi_{\pi_1} \otimes \cdots \otimes \Pi_{\pi_s} \otimes D(n/\pi_1 \ldots \pi_s),$$

where $\pi_1, \ldots, \pi_s$ *are powers of primes and their product* $\pi_1 \ldots \pi_s$ *divides* n, *with the exception of the case* $\Pi_2 \otimes \cdots \otimes \Pi_2 \otimes D(1)$. [3]

The simplest form of an Ad-group is $G = D(n)$. The MAD-group corresponding to this Ad-group gives the Cartan decomposition of $gl(n,\mathbb{C})$.

In this article we shall focus on the other extremal case, namely on the Ad-group

$$\Pi_n \otimes D(1) = \Pi_n.$$

The corresponding fine grading decomposes $gl(n,\mathbb{C})$ into a sum of n^2 one–dimensional subspaces [1, 9]

$$\Gamma_\Pi : \quad gl(n,\mathbb{C}) = \bigoplus_{(r,s)\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathcal{L}_{rs}, \tag{6}$$

where $\mathcal{L}_{rs} = \mathbb{C}X_{rs}$ with X_{rs} being the basis elements of $gl(n,\mathbb{C})$ representing n^2 cosets of Π_n with respect to its center $\{\omega^l | l \in \mathbb{Z}_n\}$:

$$X_{rs} = Q^r P^s.$$

Their commutators [4]

$$[X_{rs}, X_{r's'}] = Q^r P^s Q^{r'} P^{s'} - Q^{r'} P^{s'} Q^r P^s = (\omega^{sr'} - \omega^{rs'})X_{r+r',s+s'} \tag{7}$$

clearly satisfy the grading property with the index set I being the Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. The binary correspondence $((r,s),(r',s')) \mapsto (r+r', s+s')$, $0 \neq [\mathcal{L}_{rs}, \mathcal{L}_{r's'}] \subseteq \mathcal{L}_{r+r',s+s'}$ is the group multiplication in $\mathbb{Z}_n \times \mathbb{Z}_n$ written additively modulo n.

The corresponding grading of $sl(n,\mathbb{C})$ contains $n^2 - 1$ subspaces

$$sl(n,\mathbb{C}) = \bigoplus_{(r,s)\neq(0,0)} \mathcal{L}_{rs},$$

since $Tr(X_{rs}) = 0$ except $r = s = 0$.

4. SYMMETRIES OF THE FINE GRADINGS Γ_Π

In this section we are going to study the symmetries of the fine gradings (6) of $gl(n,\mathbb{C})$. From the previous section we know that they are induced by the Pauli group $\Pi_n \subset GL(n,\mathbb{C})$.

Generally, the symmetry group or the automorphism group Aut $\Gamma \subset$ Aut $\mathcal{L}$ of the grading (1) consists of those automorphisms ϕ of $\mathcal{L}$ which permute the components of (1),

$$\phi\mathcal{L}_i = \mathcal{L}_{\overline{\phi}(i)}.$$

Here $\overline{\phi} : I \to I$ is a permutation of the elements of I, so we have a permutation representation Δ_Γ of Aut Γ,

$$\bar{\phi} = \Delta_\Gamma(\phi), \quad \phi \in \text{Aut } \Gamma.$$

The kernel of Δ_Γ is the stabilizer of Γ in Aut Γ,

$$\text{Stab } \Gamma = \ker \Gamma = \{\phi \in \text{Aut } \mathcal{L} \mid \phi\mathcal{L}_i = \mathcal{L}_i \ \forall i \in I\}.$$

It is a normal subgroup of Aut Γ with quotient group isomorphic to the group of permutations of I,

$$\text{Aut } \Gamma/\text{Stab } \Gamma \simeq \Delta_\Gamma \text{ Aut } \Gamma.$$

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[3] In this case there is an outer automorphism in the MAD–group.

[4] Henceforth the index k in ω_k, P_k, Q_k as well as explicit notation (mod n) will be omitted.

For fine gradings Stab $\Gamma = \mathcal{G}$. The *symmetry group* Aut Γ is by definition [1] the normalizer of $\mathcal{G}$ in Aut $gl(n, \mathbb{C})$:

$$\mathcal{N}(\mathcal{G}) = \text{Aut } \Gamma = \{\phi \in \text{Aut } gl(n, \mathbb{C}) \mid \phi \mathcal{G} \phi^{-1} \subset \mathcal{G}\}.$$

Why do we look for the symmetry group Aut Γ ? We know that

(1) elements of Aut Γ / Stab Γ permute the grading subspaces,
(2) given a grading subspace, the action of Aut Γ / Stab Γ will yield
    some other grading subspaces.

So its knowledge may give us the way to construct the grading decomposition (1) from one or a small number of starting subspaces. Aut Γ / Stab Γ may also be valuable as a symmetry of the contraction equations which enables to lower their number and so simplify their solution.

Let us denote the MAD–group Ad_{Π_n} by $\mathcal{P}_n$. It is an Abelian subgroup of Aut $gl(n, \mathbb{C}) = GL(n^2, \mathbb{C})$ with generators Ad_P, Ad_Q,

$$\mathcal{P}_n = \{\text{Ad}_{Q^i P^j} \mid (i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n\}.$$

It is obvious that n^2 elements of $\mathcal{P}_n$ stabilize the grading: namely, taking the generators Ad_P, Ad_Q, one has

$$\text{Ad}_P X_{rs} = PQ^r P^s P^{-1} = \omega^r X_{rs}, \quad \text{Ad}_Q X_{rs} = QQ^r P^s Q^{-1} = \omega^{-s} X_{rs}$$

and $\mathcal{P}_n = \text{Stab } \Gamma_\Pi$ since $\mathcal{P}_n$ is maximal.

In order to describe the quotient group $\mathcal{N}(\mathcal{P}_n)/\mathcal{P}_n$ we note that its elements are classes of equivalence in $\mathcal{N}(\mathcal{P}_n)$ given by

$$\phi \sim \psi \quad \text{if and only if} \quad \phi \psi^{-1} \in \mathcal{P}_n.$$

Let $\phi \sim \psi$, i.e. $\phi \psi^{-1} = \beta$ for some $\beta \in \mathcal{P}_n$. Using the commutativity of $\mathcal{P}_n$ we have

$$\phi^{-1} \alpha \phi = \psi^{-1} \beta^{-1} \alpha \beta \psi = \psi^{-1} \alpha \psi$$

for any $\alpha \in \mathcal{P}_n$. On the other hand, let $\phi^{-1} \alpha \phi = \psi^{-1} \alpha \psi$ for any $\alpha \in \mathcal{P}_n$. Then $\phi \psi^{-1}$ commutes with every element in $\mathcal{P}_n$ and therefore $\phi \psi^{-1} \in \mathcal{P}_n$. It means that

$$\phi \sim \psi \quad \text{if and only if} \quad \phi^{-1} \alpha \phi = \psi^{-1} \alpha \psi \quad \text{for any} \quad \alpha \in \mathcal{P}_n.$$

Since the group $\mathcal{P}_n$ has only two generators Ad_P and Ad_Q, the previous condition can be rewritten

$$\phi \sim \psi \quad \text{if and only if} \quad \phi^{-1}\text{Ad}_P\phi = \psi^{-1}\text{Ad}_P\psi \quad \text{and} \quad \phi^{-1}\text{Ad}_Q\phi = \psi^{-1}\text{Ad}_Q\psi. \quad (8)$$

If ϕ belongs to the normalizer $\mathcal{N}(\mathcal{P}_n)$, then there exist elements a, b, c, d in the cyclic group $\mathbb{Z}_n$ such that

$$\phi^{-1}\text{Ad}_Q\phi = \text{Ad}_{Q^a P^b} \quad \text{and} \quad \phi^{-1}\text{Ad}_P\phi = \text{Ad}_{Q^c P^d}.$$

Thus to any equivalence class a quadruple of indices is assigned. Denote this assignment by Φ. According to (8), quadruples assigned to distinct classes are different. We shall see that it is convenient to write the quadruple as a matrix

$$\Phi(\phi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with entries from} \quad \mathbb{Z}_n.$$

Suppose that the equivalence classes containing the automorphisms ϕ_1 and ϕ_2 correspond to the quadruples a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2, respectively. Computing the quadruple assigned to the composition $\phi_1 \phi_2$

$$(\phi_1\phi_2)^{-1}\text{Ad}_Q(\phi_1\phi_2) = (\phi_2^{-1}\text{Ad}_Q\phi_2)^{a_1}(\phi_2^{-1}\text{Ad}_P\phi_2)^{b_1} =$$

$$(\text{Ad}_{Q^{a_2} P^{b_2}})^{a_1}(\text{Ad}_{Q^{c_2} P^{d_2}})^{b_1} = \text{Ad}_{Q^{a_2 a_1} P^{b_2 a_1}}\text{Ad}_{Q^{c_2 b_1} P^{d_2 b_1}} = \text{Ad}_{Q^{a_1 a_2 + b_1 c_2} P^{a_1 b_2 + b_1 d_2}},$$

$$(\phi_1\phi_2)^{-1}\text{Ad}_P(\phi_1\phi_2) = \text{Ad}_{Q^{a_1 a_2 + b_1 c_2} P^{a_1 b_2 + b_1 d_2}}$$

we see that to the automorphism $\phi_1\phi_2$ the product matrix is assigned,

$$\Phi(\phi_1\phi_2) = \Phi(\phi_1)\Phi(\phi_2).$$

Thus Φ is an injective homomorphism of the quotient group $\mathcal{N}(\mathcal{P}_n)/\mathcal{P}_n$.

Let $\phi \in \mathcal{N}(\mathcal{P}_n)$ be an *inner* automorphism, say Ad_A with the corresponding matrix

$$\Phi(\phi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with entries from } \mathbb{Z}_n.$$

Then

$$\mathrm{Ad}_A^{-1}\mathrm{Ad}_Q\mathrm{Ad}_A = \mathrm{Ad}_{A^{-1}QA} = \mathrm{Ad}_{Q^aP^b} \quad \text{implies} \quad A^{-1}QA = \mu Q^a P^b, \qquad (9)$$

$$\mathrm{Ad}_A^{-1}\mathrm{Ad}_P\mathrm{Ad}_A = \mathrm{Ad}_{A^{-1}PA} = \mathrm{Ad}_{Q^cP^d} \quad \text{implies} \quad A^{-1}PA = \nu Q^c P^d \qquad (10)$$

for some $\mu, \nu \in \mathbb{C}^*$. Multiplying the equations (9) and (10) by PA and QA from the right, respectively, and using the relation $PQ = \omega QP$, we obtain

$$PQA = \mu\nu AQ^c P^d Q^a P^b = \omega^{ad}\mu\nu Q^{a+c} P^{b+d}$$

and

$$QPA = \mu\nu AQ^a P^b Q^c P^d = \omega^{bc}\mu\nu Q^{a+c} P^{b+d}$$

Since $PQA = \omega QPA$ we obtain the identity

$$\omega^{ad-1} = \omega^{bc}, \quad \text{i.e.} \quad ad - 1 = bc \ (mod \ n),$$

hence

$$\det \Phi(\phi) = 1.$$

A simple computation further shows that for this inner automorphism one has

$$A^{-1}X_{rs}A = \rho X_{r's'},$$

where $|\rho| = 1$ and

$$(r', s') = (r, s)\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (11)$$

Consider now the outer automorphism $Out_I \ X := -X^T$. Because of

$$(\mathrm{Out}_I)^{-1}\mathrm{Ad}_Q\mathrm{Out}_I = \mathrm{Ad}_{(Q^{-1})^T} = \mathrm{Ad}_{Q^{-1}} \in \mathcal{P}_n,$$

$$(\mathrm{Out}_I)^{-1}\mathrm{Ad}_P\mathrm{Out}_I = \mathrm{Ad}_{(P^{-1})^T} = \mathrm{Ad}_P \in \mathcal{P}_n,$$

the automorphism Out_I belongs to the normalizer $\mathcal{N}(\mathcal{P}_n)$. The matrix corresponding to Out_I is

$$\Phi(\mathrm{Out}_I) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \det \Phi(\mathrm{Out}_I) = -1.$$

Note that

$$\mathrm{Out}_I \ X_{rs} = -\omega^{-rs} X_{r,-s}.$$

corresponds to the permutation of indices

$$(r, s) \mapsto (r, -s).$$

Any other outer automorphism ϕ from $\mathcal{N}(\mathcal{P}_n)$ is the composition of Out_I and an inner automorphism from $\mathcal{N}(\mathcal{P}_n)$ and thus $\det \Phi(\phi) = -1$. We conclude with

Proposition 6. Φ *is an injective homomorphism of* $\mathcal{N}(\mathcal{P}_n)/\mathcal{P}_n$ *into the group*

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_n, \ ad - bc = \pm 1 \mod n, \right\}.$$

The group H contains as its subgroup the group of matrices with determinant $+1$. This group is usually denoted by $SL(2, \mathbb{Z}_n)$. Note that $\mathbb{Z}_n$ is a field iff n is prime. Clearly

$$H = SL(2, \mathbb{Z}_n) \bigcup \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \ SL(2, \mathbb{Z}_n).$$

Let us briefly show that for any $n \in \mathbb{N}$, the group $SL(2, \mathbb{Z}_n)$ is generated by two matrices

$$A = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \quad \text{and} \quad B = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$

In order to show it, we have to realize that in the ring $\mathbb{Z}_n$, the matrices A and B are of orders n and 4, respectively. The matrix $C = A^T$ is generated by A and B:

$$C = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)^{n-1} \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)^3.$$

Moreover, any matrix of $SL(2, \mathbb{Z}_n)$ satisfies

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \left(\begin{array}{cc} a & b \\ c-a & d-b \end{array} \right) = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \left(\begin{array}{cc} a-c & b-d \\ c & d \end{array} \right). \tag{12}$$

Now recall that Euclid's algorithm for finding the greatest common divisor of integers is based on the trivial fact that $gcd(x, y) = gcd(x - y, y)$. By several repetitions of this rule, where we replace the pair of non–negative integers $\{x, y\}, x \geq y$, by another pair of non–negative integers $\{x - y, y\}$, the Euclid's algorithm gives finally a pair of integers, where one of them is 0 and the other is $gcd(x, y)$.

Denote $gcd(a, c) = s$. Then by suitable applications of (12) we obtain

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = A^{k_1} C^{l_1} \dots A^{k_p} C^{l_p} T$$

where $k_1, l_1, \dots, k_p, l_p \in \mathbb{N}_0$ and T is a matrix with det $T = 1$, of the form

$$T = \left(\begin{array}{cc} s & t \\ 0 & u \end{array} \right) \quad \text{or} \quad T = \left(\begin{array}{cc} 0 & v \\ s & w \end{array} \right).$$

But any matrix T of such form is a product of several matrices A,B, and C, since

$$\left(\begin{array}{cc} s & t \\ 0 & u \end{array} \right) = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)^{s(t-1)} \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)^u \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)^s$$

and

$$\left(\begin{array}{cc} 0 & v \\ s & w \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left(\begin{array}{cc} -s & -w \\ 0 & v \end{array} \right).$$

This proves that $SL(2, \mathbb{Z}_n)$ has two generators, the matrices A and B.

Now we shall present two elements of the normalizer — special unitary $n \times n$-matrices inducing Ad-actions which represent generating elements of $SL(2, \mathbb{Z}_n)$.

Example 7. Since the matrices Q and P have the same spectra, they are similar with a similarity matrix S such that $S^{-1}PS = Q$. Such S is not determined uniquely. We choose for the matrix S the Sylvester matrix defined as follows

$$S_{ij} = \omega^{-ij}, \quad \text{for} \ i, j \in \mathbb{Z}_n.$$

It is easy to verify that S^2 is a parity operator

$$S_{ij}^2 = \delta_{i,-j} \ \text{for} \ i, j \in \mathbb{Z}_n \ \text{such that} \ S^4 = I.$$

Note that the indices (and operations on them) are always considered to be elements of the ring $\mathbb{Z}_n$. Let us verify that Ad_S belongs to the normalizer. Note that S is a symmetric matrix and therefore $Q = Q^T = (S^{-1}PS)^T = SP^TS^{-1} = SP^{-1}S^{-1}$, which implies $S^{-1}QS = P^{-1}$. Now we can easily check the conditions on Ad_S to be in the normalizer:

$$(Ad_S)^{-1} Ad_Q Ad_S = Ad_{S^{-1}QS} = Ad_{P^{-1}} \in \mathcal{P}_n$$

and

$$(Ad_S)^{-1} Ad_P Ad_S = Ad_{S^{-1}PS} = Ad_Q \in \mathcal{P}_n$$

By (9) and (10) the matrix corresponding to Ad_S is

$$\Phi(Ad_S) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example 8. For this example we shall use the similarity of matrices P and PQ. Put $\varepsilon = 1$, if n is odd, and $\varepsilon = \sqrt{\omega}$, if n is even. Denote by

$$D = \mathrm{diag}(d_0, d_1, \ldots, d_{n-1}), \quad \text{where } d_j = \varepsilon^{-j} \omega^{-\binom{j}{2}}, \quad \text{for } j \in \mathbb{Z}_n$$

It is easy to see that $Q = D^{-1}QD$ and $PQ = \varepsilon D^{-1}PD$; it implies

$$(Ad_D)^{-1} Ad_Q Ad_D = Ad_{D^{-1}QD} = Ad_Q \in \mathcal{P}_n$$

and

$$(Ad_D)^{-1} Ad_P Ad_D = Ad_{D^{-1}PD} = Ad_{PQ} \in \mathcal{P}_n$$

which means that Ad_D belongs to the normalizer. By (9) and (10) the matrix assigned to Ad_D is therefore

$$\Phi(Ad_D) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The homomorphism Φ maps three elements of the normalizer Out_I, Ad_S, and Ad_D into three matrices generating the whole group H. This observation together with the Proposition 6 gives us

Theorem 10. *The factor group $\mathcal{N}(\mathcal{P}_n)/\mathcal{P}_n$ is isomorphic to the group*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_n, \ ad - bc = \pm 1 \bmod n \right\} = SL(2, \mathbb{Z}_n) \bigcup \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) SL(2, \mathbb{Z}_n).$$

The direct consequence of Theorem 10 is

Corollary 11. *The normalizer $\mathcal{N}(\mathcal{P}_n)$ of the group $\mathcal{P}_n$ is generated by*

$$Out_I, Ad_S, Ad_D, Ad_Q \quad \text{and} \quad Ad_P$$

If n is prime, i.e. $\mathbb{Z}_n$ is a field, we can use the Bruhat decomposition of $SL(2, \mathbb{Z}_n)$ and explicitly describe the normalizer. It enables us to count the number of its elements. The group $SL(2, \mathbb{Z}_n)$ is the union of two disjoint sets

$$\left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \mid a \in \mathbb{Z}_n, b \in \mathbb{Z}_n^* \right\}$$

and

$$\left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid a, c \in \mathbb{Z}_n, b \in \mathbb{Z}_n^* \right\}.$$

Corollary 12. *Let n be a prime. Any element of the normalizer $\mathcal{N}(\mathcal{P}_n)$ has the form*

$$Ad_A \text{ or } Out_I Ad_A, \text{ where } A = D^i M^j Q^k P^l \text{ or } A = D^i M^j S D^s Q^k P^l$$

and M is a permutation matrix described in the Appendix. The normalizer $\mathcal{N}(\mathcal{P}_n)$ has therefore $2(n^2 - 1)n$ elements for an odd prime n and 24 elements for $n = 2$ (here the outer automorphism does not play any role)

Summarizing, for any natural number n we described the generators of the normalizer $\mathcal{N}(\mathcal{P}_n)/\mathcal{P}_n$. For n prime we moreover were able to determine the cardinality of normalizer, using the Bruhat decomposition of the group $SL(2, \mathbb{Z}_n)$. For the explicit description of the special n–dimensional representation of $SL(2, \mathbb{Z}_n)$, where n is prime of the form $n = 4K \pm 1$, see [20].

5. GRADED CONTRACTIONS OF $sl(3, \mathbb{C})$

In this section we want to illustrate how the explicit knowledge of the normalizer of a fine grading can simplify and, indeed, bring further insight into the structure of the problem of finding all graded contractions of $sl(3, \mathbb{C})$. Practically one needs to solve a system of 48 quadratic equations involving 28 contraction parameters. The normalizer is the symmetry group of that system. It turns out that there are only two orbits of the normalizer among the 48 quadratic equations. For more conventional approach to this problem see [11, 21].

Let $\mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i$ be a grading decomposition of a Lie algebra with the commutator $[\cdot, \cdot]$. Definition of the contracted commutator of the algebra involves the contraction parameters ε_{ij} for $i, j \in I$ and the old commutator. The new bilinear mapping of the form

$$[x, y]_{new} := \varepsilon_{ij}[x, y] \qquad \text{for all } x \in \mathcal{L}_i, \ y \in \mathcal{L}_j$$

is a commutator on the same vector space $\mathcal{L}$.

To satisfy antisymmetry of the commutator we have to choose $\varepsilon_{ij} = \varepsilon_{ji}$. To satisfy the Jacobi identity one has to solve a system of quadratic equations for the unknown contraction parameters ε_{ij}.

Let us illustrate this problem on the graded algebra $sl(3, \mathbb{C}) = \bigoplus_{(i,j) \neq (0,0)} \mathcal{L}_{ij}$, which has 8 one-dimensional graded subspaces $\mathcal{L}_{ij} = \mathbb{C}X_{ij}$, where $0 \leq i, j \leq 2$. For example, for the triple of vectors $X_{(0,1)}$, $X_{(0,2)}$ and $X_{(1,0)}$ the Jacobi identity has the form

$$[X_{(0,1)}, [X_{(0,2)}, X_{(1,0)}]_{new}]_{new} + \text{cyclically } = 0.$$

The commutation relations 7 give us

$$\varepsilon_{(02)(10)}\varepsilon_{(01)(12)}(\omega - 1)(\omega^2 - 1)X_{(1,0)} + \varepsilon_{(10)(01)}\varepsilon_{(02)(11)}(1 - \omega)(\omega^2 - 1)X_{(1,0)} = 0$$

and therefore

$$\varepsilon_{(02)(10)}\varepsilon_{(01)(12)} - \varepsilon_{(10)(01)}\varepsilon_{(02)(11)} = 0 \tag{13}$$

For all possible triples of basis elements $X_{(i,j)}$ we have to write similar equations. There are $\binom{8}{3} = 56$ triples. Since triples of the form $X_{(a,b)}$, $X_{(c,d)}$ and $X_{(e,f)}$, with $a + c + e = 0 \ (mod \ 3)$ and $b + d + f = 0 \ (mod \ 3)$ satisfy $[X_{(a,b)}, [X_{(c,d)}, X_{(e,f)}]] = 0$, we have in fact only 48 equations.

The Jacobi identity for the triple $X_{(0,1)}$, $X_{(1,0)}$ and $X_{(1,1)}$, is the equality

$$\varepsilon_{(10)(11)}\varepsilon_{(01)(21)} - \varepsilon_{(11)(01)}\varepsilon_{(10)(12)} = 0 \ . \tag{14}$$

The triple of indices $(0, 1)$, $(1, 0)$ and $(1, 1)$ is distinguished by the property that the indices of any epsilon appearing in (14) are linearly independent over the field $\mathbb{Z}_3$. Quite different are the indices in (13). There the pair of indices $(0, 1)$ and $(0, 2)$ is linearly dependent over the field $\mathbb{Z}_3$. These two cases exhaust all distinct possibilities for the choice of triples in the Jacobi identity.

Consider now the mappings on the index set I defined by (11)

$$(i, j) \mapsto (i, j)A \qquad \text{where } A \in SL(2, \mathbb{Z}_3)$$

Applying such a mapping with a fixed matrix A to the indices occurring in equation (13) we obtain a new equation corresponding to the Jacobi identity for another triple of grading subspaces. If we gradually apply all 24 matrices from $SL(2, \mathbb{Z}_3)$ to the equations (13) and (14), we obtain all 48 quadratic equations which should be satisfied. In this way the symmetries of the system of equations are directly seen.

Appendix. Consider $n \times n$ matrices M_s defined by

$$(M_s)_{i,j} = \delta_{i,sj} \quad \text{for} \quad i, j \in Z_n,$$

where n is a prime and $s \in Z_n^* = Z_n \backslash \{0\}$. Since Z_n is a field, the equation $is = j$ has one solution i for fixed s and j and similarly one solution j for fixed s and i. It means that each matrix M_s has exactly one 1 in each column and row; M_s are thus permutation matrices. Moreover, M_1 is the unit matrix and $M_s M_t = M_{st}$ holds:

$$(M_s M_t)_{ij} = \sum_{k=0}^{n-1} (M_s)_{ik} (M_t)_{kj} = \sum_{k=0}^{n-1} \delta_{i,sk} \delta_{k,tj} = \delta_{i,jst} = (M_{st})_{ij},$$

hencet the matrices M_s, $s = 1, 2, \ldots, n-1$ form a representation of a multiplicative Abelian group. This group has only one generator, because for any finite field $(F, +, \cdot)$ the group $(F^* = F \backslash \{0\}, \cdot)$ is cyclic. In particular for Z_n^* there exists an element, say a, such that $Z_n^* = \{a^i \mid i = 1, 2, \ldots, n-1\}$. Therefore $M_1, \ldots, M_{n-1}$ is a cyclic group of order $n-1$ with generator M_a and we have $M_{a^s} = M_a^s$.

We shall show that Ad_{M_s} belongs to the normalizer $\mathcal{N}(\mathcal{P}_n)$. We use the fact that M_s is a permutation matrix and therefore $M_s^{-1} = M_s^T$. Let us compute $M_s^{-1} Q M_s$ and $M_s^{-1} P M_s$:

$$(M_s^{-1} Q M_s)_{ij} = \sum_{k=0}^{n-1} (M_s^T)_{ik} (Q M_s)_{kj} = \sum_{k=0}^{n-1} \sum_{r=0}^{n1} (M_s)_{ki} Q_{kr} (M_s)_{rj} =$$

$$= \sum_{k=0}^{n-1} \sum_{r=0}^{n-1} \delta_{k,is} \omega^k \delta_{kr} \delta_{r,sj} = \sum_{k=0}^{n-1} \delta_{k,is} \omega^k \delta_{k,sj} = \delta_{ij} \omega^{si} = (Q^s)_{ij},$$

hence

$$M_s^{-1} Q M_s = Q^s \ . \tag{15}$$

Further,

$$(M_s^{-1} P M_s)_{ij} = \sum_{k=0}^{n-1} (M_s^T)_{ik} (P M_s)_{kj} = \sum_{k=0}^{n-1} \sum_{r=0}^{n-1} (M_s)_{ki} P_{kr} (M_s)_{rj} =$$

$$= \sum_{k=0}^{n-1} \sum_{r=0}^{n1} \delta_{k,is} \delta_{k,r+1} \delta_{r,sj} = \sum_{r=0}^{n-1} \delta_{is,r+1} \delta_{r,sj} = \delta_{is,js+1}$$

and noting that $\delta_{is,js+1} = \delta_{i,j+t}$, where $st = 1$ in Z_n, we have

$$M_s^{-1} P M_s = P^{s^{-1}} \ . \tag{16}$$

Equations (15) and (16) mean that all Ad_{M_s} belong to the normalizer and that the matrix assigned to M_s is

$$\Phi(M_s) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}.$$

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